RC Filter Derivation

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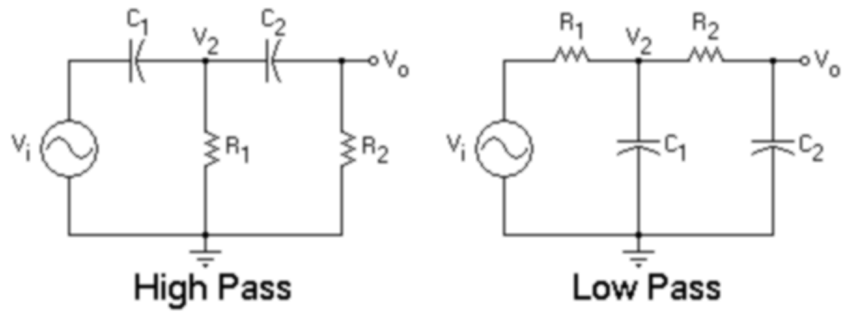

Figure 1: RC second order filters, high pass and low pass.

In this paper I will analyze the frequency response of the cascaded (second order) RC filter, high and low pass, mostly concentrating on the high pass filter (in general, the low pass filter's formulas are identical with the exception of one factor). This classic circuit is shown in Figure 1. I will start by analyzing its elemental unit, the first-order RC filter. This will be analyzed in general terms, appropriate impedances substituted, simplifying to obtain the magnitude of the voltage gain. Then the second order filter will be analyzed in a similar manner. The two cases will be compared to illustrate their differences. Finally, frequency response will analyzed and cutoff frequencies determined.

First Order Filter

Either high or low pass can be evaluated, but a more general statement can be made using unspecified impedances Z_1 and Z_2 as in Figure 2.

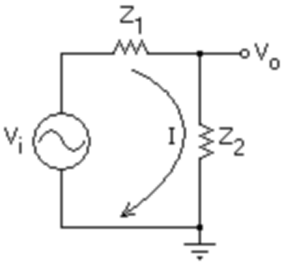

Figure 2: First order impedance divider (not necessarily a filter).

The total impedance is $Z_1 + Z_2$, so the current is $I = \frac{V_i}{Z_1+Z_2}$ and the output voltage $V_o = IZ_2$ or,

$$V_o = V_i \frac{Z_2}{Z_1 + Z_2} \tag{1}$$

Let $Z_1 = X_C = \frac{1}{j\omega C}$ and $Z_2 = R$, where X_C is the reactance of a capacitance and R is a resistance. The circuit is now a high-pass filter. Since it appears in every equation that follows, the substitution $\beta = \omega RC$ will be used throughout this paper. Note, also, that ω is the radial frequency, so the ordinary frequency f $= {}^{\omega}\!/_{2\pi}$.

$$\begin{aligned}
\frac{V_o}{V_i}_{(HP^1)} &= \frac{R}{X_C + R} = \frac{R}{\frac{1}{j\omega C} + R} \\
&= \frac{R}{\frac{j\beta+1}{j\omega C}} = \frac{j\beta}{j\beta + 1} \\
&= \frac{(j\beta)(1-j\beta)}{(1+j\beta)(1-j\beta)} \\
&= \frac{\beta^2 + j\beta}{\beta^2 + 1} && (2) \\
\left|\frac{V_o}{V_i}\right|_{(HP^1)} &= \sqrt{\frac{(\beta^2)^2 + (\beta)^2}{(\beta^2+1)^2}} \\
&= \frac{\sqrt{\beta^4 + \beta^2}}{\beta^2 + 1} \\
&= \frac{\beta\sqrt{\beta^2+1}}{\beta^2+1} \\
&= \frac{\beta}{\sqrt{\beta^2+1}} && (3)
\end{aligned}$$

The low pass formula is easily derived similarly, yielding:

$$\left|\frac{V_o}{V_i}\right|_{(LP^1)} = \frac{1}{\sqrt{\beta^2+1}} \tag{4}$$

Equation (2) gives the complex gain of the high pass filter. (3) gives the magnitude of that gain. The complex gain of the low pass filter is not provided but the magnitude is given in (4).

Second Order Filter

The filters shown in Figure 1 can be generalized by replacing resistors and capacitors in both circuits with impedances Z_1 through Z_4, as in Figure 3.

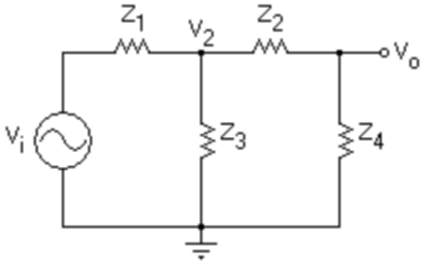

Figure 3: General second order impedance divider.

The output voltage V_o is given in terms of V_2 by (1), but the voltage at V_2 is more complicated. To solve, we find the effective impedance to the right of V_2, which consists of Z_2 and Z_4 in series, this in parallel with Z_3, or:

$$Z_{eff} = \frac{Z_3(Z_2 + Z_4)}{Z_2 + Z_3 + Z_4} \tag{5}$$

Z_{eff} forms a voltage divider with Z_1, so the voltage V_2 is given by:

$$V_2 = V_i \frac{Z_{eff}}{Z_1 + Z_{eff}}$$

Substituting this into equation (1) (switching the appropriate variables), expanding and simplifying gives:

$$\begin{aligned}
\frac{V_o}{V_i} &= \frac{Z_4}{Z_2+Z_4}\frac{Z_{eff}}{Z_1+Z_{eff}} \\
&= \frac{Z_4}{Z_2+Z_4}\frac{\frac{Z_3(Z_2+Z_4)}{Z_2+Z_3+Z_4}}{Z_1+\frac{Z_3(Z_2+Z_4)}{Z_2+Z_3+Z_4}} \\
&= \frac{\frac{Z_3Z_4}{Z_2+Z_3+Z_4}}{\frac{Z_1(Z_2+Z_3+Z_4)+Z_3(Z_2+Z_4)}{Z_2+Z_3+Z_4}} \\
&= \frac{Z_3Z_4}{Z_1(Z_2+Z_3+Z_4)+Z_3(Z_2+Z_4)} \qquad (6)
\end{aligned}$$

Again substituting for Zs, this time including an impedance factor γ in the second stage components Z_2 and Z_4, gives $Z_1 = \frac{1}{j\omega C}$, $Z_2 = \frac{\gamma}{j\omega C}$, $Z_3 = R$ and $Z_4 = \gamma R$ for the high pass filter.

$$\begin{aligned}
\frac{V_o}{V_i}_{(HP^2)} &= \frac{(R)(\gamma R)}{\frac{1}{j\omega C}(\frac{\gamma}{j\omega C}+R+\gamma R)+R(\frac{\gamma}{j\omega C}+\gamma R)} \\
&= \frac{\gamma R^2}{\frac{\gamma}{(j\omega C)^2}+\frac{R}{j\omega C}+\frac{\gamma R}{j\omega C}+\frac{\gamma R}{j\omega C}+\gamma R^2} \\
&= \frac{\gamma R^2}{\frac{\gamma+R(j\omega C)+2\gamma R(j\omega C)+\gamma R^2(j\omega C)^2}{(j\omega C)^2}} \\
&= \frac{-\gamma\beta^2}{\gamma+j\beta+2j\gamma\beta-\gamma\beta^2} \\
&= \frac{\beta^2}{\beta^2-1-j({}^{1}\!/_{\gamma}+2)\beta} \\
&= \frac{(\beta^2)[\beta^2-1+j({}^{1}\!/_{\gamma}+2)\beta]}{[\beta^2-1-j({}^{1}\!/_{\gamma}+2)\beta][\beta^2-1+j({}^{1}\!/_{\gamma}+2)\beta]} \\
&= \frac{\beta^4-\beta^2+j({}^{1}\!/_{\gamma}+2)\beta^3}{(\beta^2-1)^2+({}^{1}\!/_{\gamma}+2)^2\beta^2} \\
&= \frac{\beta^4-\beta^2+j({}^{1}\!/_{\gamma}+2)\beta^3}{\beta^4-2\beta^2+1+({}^{1}\!/_{\gamma^2}+{}^{4}\!/_{\gamma}+4)\beta^2} \\
&= \frac{\beta^4-\beta^2+j({}^{1}\!/_{\gamma}+2)\beta^3}{\beta^4+1+({}^{1}\!/_{\gamma^2}+{}^{4}\!/_{\gamma}+2)\beta^2} \qquad (7)
\end{aligned}$$

$$\begin{aligned}\left|\frac{V_o}{V_i}\right|_{(HP^2)} &= \sqrt{\frac{(\beta^4-\beta^2)^2+[({}^1\!/_\gamma+2)\beta^3]^2}{[\beta^4+1+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)\beta^2]^2}} \\ &= \frac{\sqrt{\beta^8-2\beta^6+\beta^4+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+4)\beta^6}}{\beta^4+1+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)\beta^2} \\ &= \frac{\beta^2\sqrt{\beta^4+1+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)\beta^2}}{\beta^4+1+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)\beta^2} \\ &= \frac{\beta^2}{\sqrt{\beta^4+1+({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)\beta^2}} \qquad (8)\end{aligned}$$

Equation (6) gives the general voltage gain of the impedance network, while (7) gives the full complex voltage gain for a high pass filter. The voltage magnitude is given in (8).

It can be seen by inspection that (8) tends towards zero as $\omega \to 0$, and towards unity as $\omega \to \infty$, as a high pass filter should. It can also be seen that, for very small ω, the denominator approaches unity and the gain becomes proportional to ω^2, a characteristic -40dB/decade slope, proving this filter is, in fact, second order. What is the behavior in the crossover region? I shall call γ the 'sharpness' factor. If the circuit is broken into two RC stages, it becomes clear that the second stage loads the first, particularly around sensitive frequencies, like in the crossover region. Even with rather heavy loading, the asymptotic behavior is always preserved, but the rate of approach varies considerably. If there is no loading effect, then the expected gain equals the first-order case squared, or:

$$\left|\frac{V_o}{V_i}\right| = \left(\frac{\beta}{\sqrt{\beta^2+1}}\right)^2 = \frac{\beta^2}{\beta^2+1}$$

But this can be rewritten by multiplying out under the square root,

$$= \frac{\beta^2}{\sqrt{\beta^4+2\beta^2+1}}$$

which is exactly equal to (8) when $\gamma \to \infty$, in other words, with no loading from the second stage (when its impedance is infinite). As $\gamma \to 0$, the impedance of the second stage approaches zero, so the output (for all frequencies) should approach zero. Since ${}^1\!/_{a^2}$ in the denominator becomes very large, the whole expression becomes small and tends to zero.

Some practical values for γ include unity (a straight 'RCRC' filter constructed with equal resistors and equal capacitors), and an approximation of reduced loading, $\gamma = 10$ for example, where $R_2 = 10R_1$ and $C_1 = 10C_2$. In the first case, the middle expression's coefficient $({}^1\!/_{\gamma^2}+{}^4\!/_\gamma+2)$ evaluates to 7, a somewhat wide figure, while in the latter case, it comes to 2.41, not a bad approximation (within 20%) of the ideal coefficient 2. A plot of the two curves is shown in Figure 4.

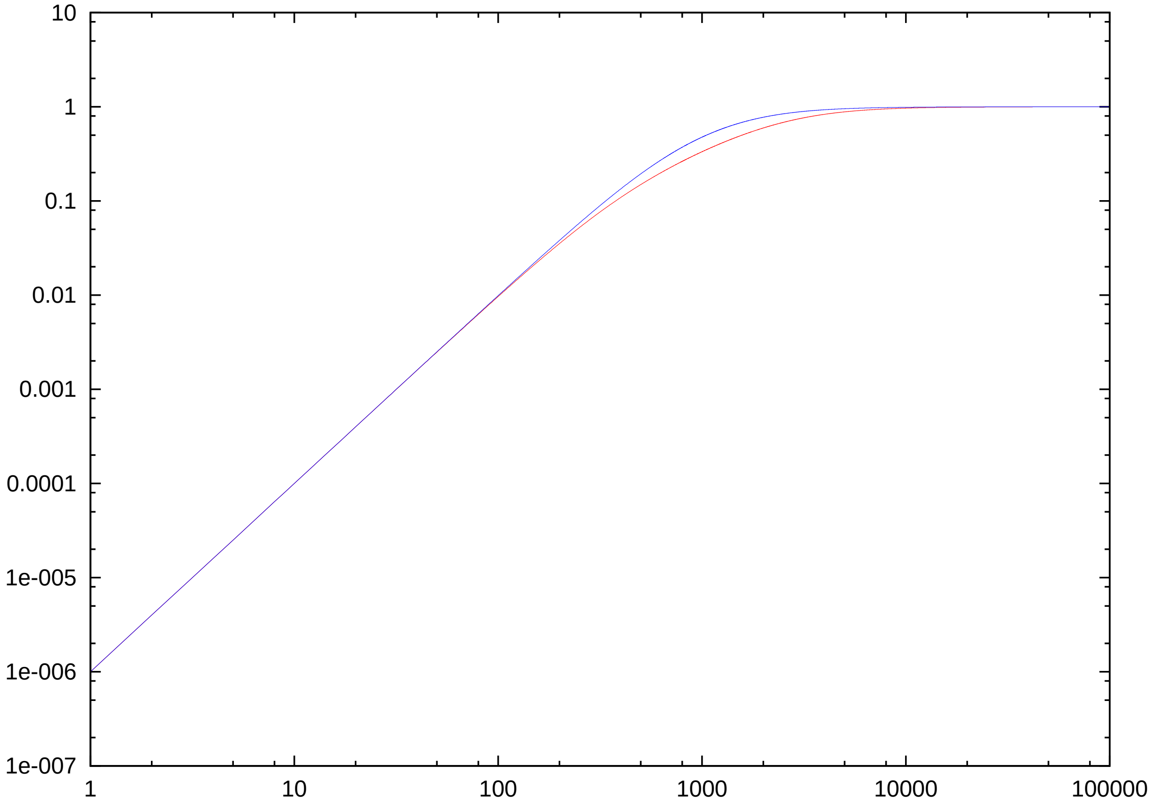

Figure 4: Plot of voltage gain (magnitude) versus angular frequency ω, for $R = 10k\Omega$ and $C = 0.1\mu F$. Blue curve: $\gamma = 10$, red curve: $\gamma = 1$.

Frequency Response

The cutoff frequency can be obtained by solving for ω and letting it equal a cutoff ratio, such as $-3dB$ $(= {}^{\sqrt{2}}\!/\!_{2})$. For the first-order high pass filter, this gives:

$$\begin{aligned}
\frac{V_o}{V_i}_{(HP^1)} = \frac{\sqrt{2}}{2} &= \frac{\beta}{\sqrt{\beta^2+1}} \\
\frac{1}{2} &= \frac{\beta^2}{\beta^2+1} \\
\beta^2 + 1 &= 2\beta^2 \\
\beta^2 &= 1 \\
\omega &= (\pm)\frac{1}{RC}
\end{aligned} \tag{9}$$

Thus obtaining the simple, familiar result (9), or converting from radians, f $= \frac{1}{2\pi RC}$. On the other hand, the second order filter gives a somewhat more complicated solution.

$$\begin{aligned}
\frac{V_o}{V_i}_{(HP^2)} = \frac{\sqrt{2}}{2} &= \frac{\beta^2}{\sqrt{\beta^4 + 1 + ({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2)\beta^2}} \\
2\beta^4 &= \beta^4 + 1 + ({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2)\beta^2 \\
\beta^4 - ({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2)\beta^2 - 1 &= 0 \\
\beta^2 &= \frac{({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2) \pm \sqrt{({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2)^2 - 4(1)(-1)}}{2(1)} \\
\omega &= (\pm)\frac{\sqrt{({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2) \pm \sqrt{({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2)^2 + 4}}}{\sqrt{2}RC} \qquad (10)
\end{aligned}$$

For a typical value of $\gamma = 1$, this gives (notice the negative implies there is an imaginary cutoff point as well):

$$\begin{aligned}
\omega &= \frac{\sqrt{7 \pm \sqrt{7^2 + 4}}}{\sqrt{2}RC} \\
\omega &\approx \frac{\sqrt{7 \pm 7.28}}{\sqrt{2}RC} \\
\omega &\approx \frac{2.67}{RC} \qquad (11)
\end{aligned}$$

For $\gamma = 10$, the cutoff frequency is:

$$\omega \approx \frac{1.66}{RC} \qquad (12)$$

In the limit as $\gamma \to \infty$, the term $({}^1\!/_{\gamma^2} + {}^4\!/_{\gamma} + 2) \to 2$ and the cutoff frequency is,

$$\omega \approx \frac{1.55}{RC} \qquad (13)$$

which is the cutoff frequency of the ideal (non-interacting) two stage filter.